Mathematics Specialist

Unit Two

2

UNIT TWO PRELIMINARY WORK

Degrees and radians

Complementary angle properties

Having reached this stage of this book it is naturally assumed that you are already familiar with the work of the previous eight chapters. It is also assumed that you are familiar with the content of Unit One of the *Mathematics Methods* course. The elements of that unit that are of particular relevance to this unit are briefly revised in this section.

Read this 'preliminary work' section and if anything is not familiar to you, and you don't understand the brief mention or explanation given here, you may need to do some further reading to bring your understanding of those concepts up to an appropriate level for this unit.

Radian measure

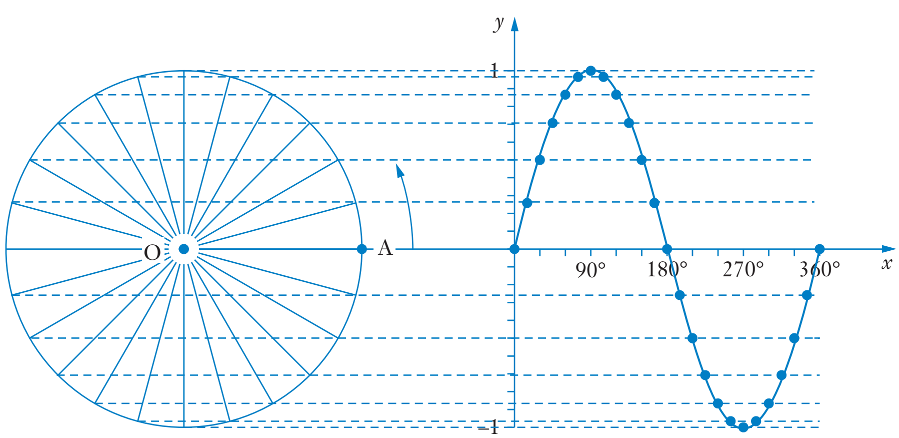
It is assumed that you are familiar with the idea of using a **radian** as a unit of measurement for angles and with the conversion:

$$\pi \text{ radians} = 180°$$

In this unit, assume the angle measure is in radians unless degrees are clearly indicated.

Unit circle definitions of $y = \sin x$, $y = \cos x$ and $y = \tan x$

With point O as the origin and point A as a point moving around a circle of unit radius and centre at O, we define the sine of the angle that AO makes with the positive x-axis as the y-coordinate of A. From this definition we obtain the graph of the function $y = \sin x$, with degrees as our unit of measure, as shown below right.



If, having completed one rotation of the circle, we were to continue moving point A around the circle the graph would repeat itself, as shown on the next page for three rotations.

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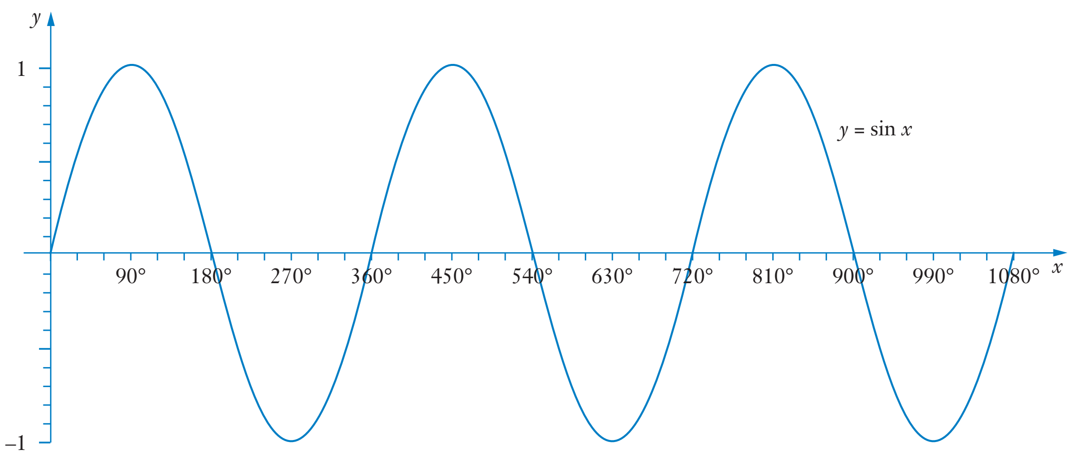
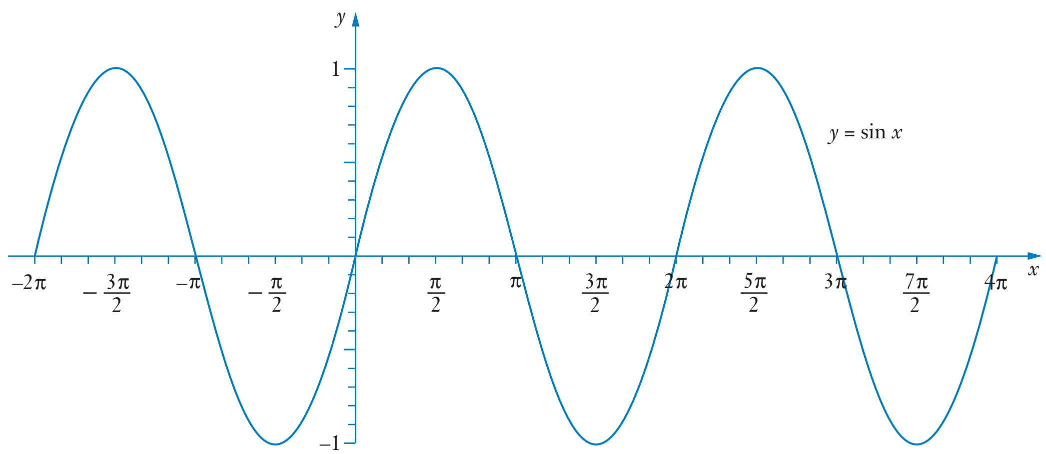


Alternatively angles could be shown in radians and negative angles could also be included, as shown below for $-2\pi \le x \le 4\pi$.



Whilst the graph shown above is for $-2\pi \le x \le 4\pi$ this restriction is made purely due to page width limitations. The reader should consider the graph of $y = \sin x$ continuing indefinitely to the left and the right.

Note the following:

- The graph of $y = \sin x$ repeats itself every 2π radians (or 360°).
  We say that the sine function is **periodic**, with **period** 2π.
  Thus $\sin(x \pm 2\pi) = \sin x$.
- We also say that the graph performs one **cycle** each period.
  Thus $y = \sin x$ performs one cycle in 2π radians (or 360°).
- $-1 \le \sin x \le 1$.
- If we consider the above graph to have a 'mean' y-coordinate of $y = 0$ then the graph has a maximum value 1 above this mean value and a minimum value 1 below it. We say that $y = \sin x$ has an **amplitude** of 1.
- $\sin(-x) = -\sin x$. (Functions for which $f(-x) = -f(x)$ are called *odd* functions and are unchanged under a 180° rotation about the origin.)

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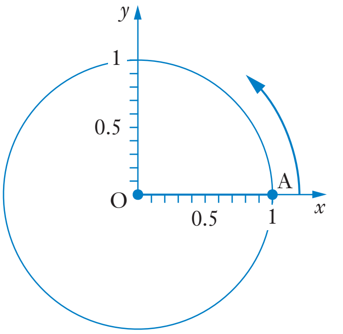
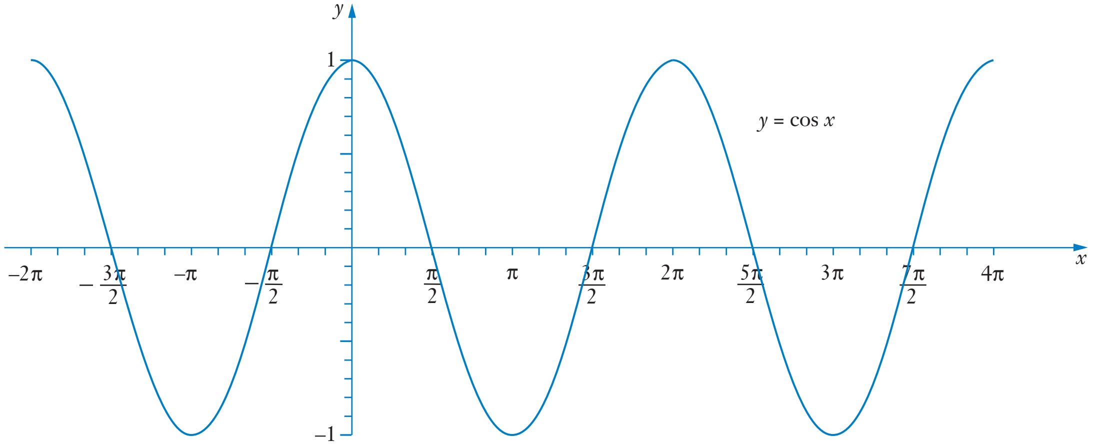
Similar considerations of the ***x*-coordinate** of point A as it moves around the unit circle gives the graph of

$$\boldsymbol{y = \cos x}$$

shown below for $-2\pi \le x \le 4\pi$.





Note the following:

- The cosine function is **periodic**, with **period 2π**. Thus $\cos(x \pm 2\pi) = \cos x$.
- $y = \cos x$ performs one **cycle** in 2π radians (or 360°).
- Note that $-1 \le \cos x \le 1$.
- The graph of $y = \cos x$ has an **amplitude** of 1.
- Note that $\cos(-x) = \cos x$. (Functions for which $f(-x) = f(x)$ are called *even* functions and are unchanged under a reflection in the y-axis.)
- If the above graph of $y = \cos x$ is moved $\frac{\pi}{2}$ units right, parallel to the x axis, it would then be the same as the graph of $y = \sin x$. We say that $\sin x$ and $\cos x$ are $\frac{\pi}{2}$ out of **phase** with each other.

  It follows that $\cos x = \sin\left(x + \frac{\pi}{2}\right)$ and $\sin x = \cos\left(x - \frac{\pi}{2}\right)$.

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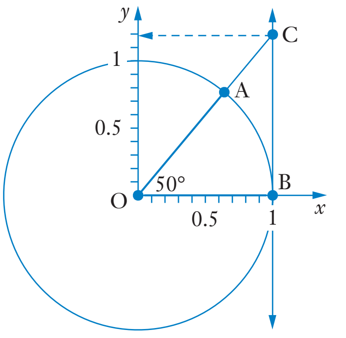
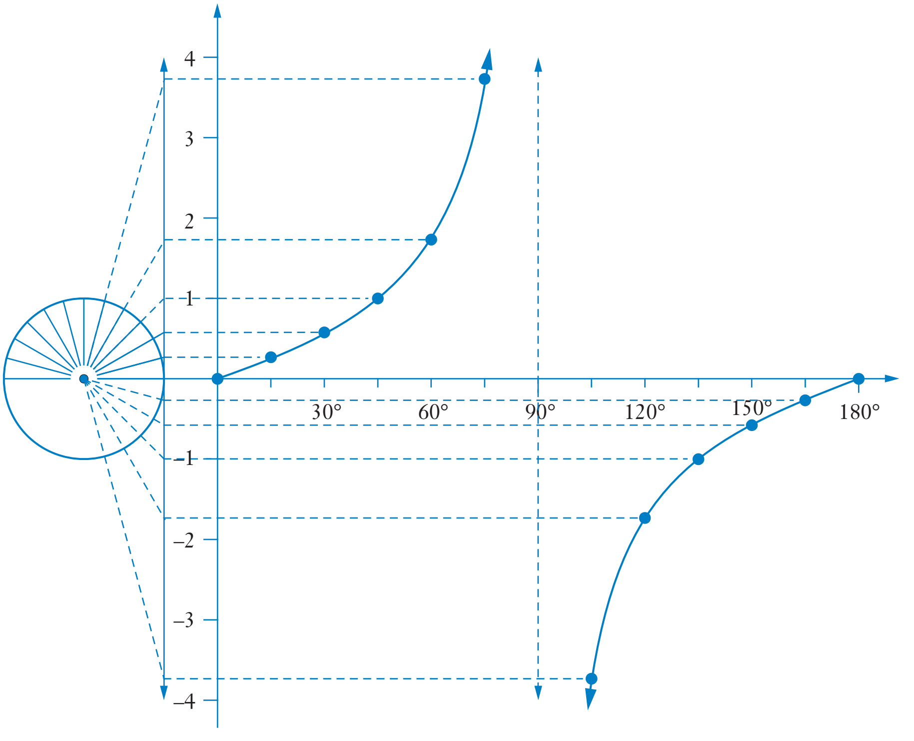
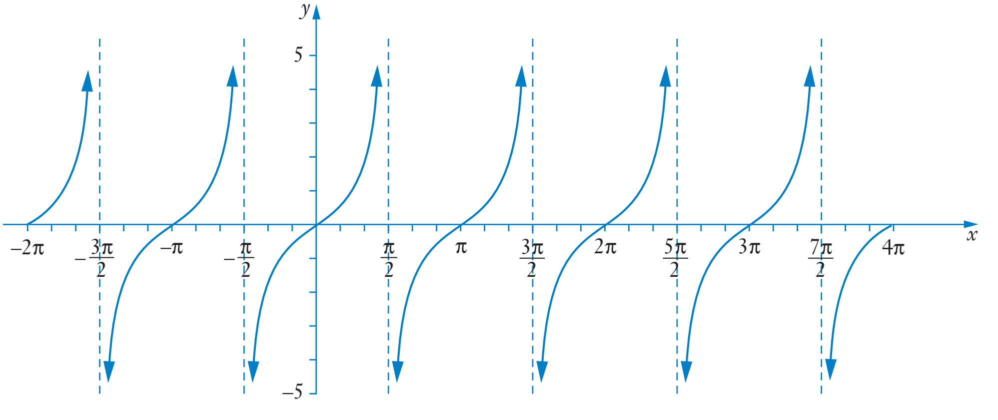
Either by considering the relationship $\tan x = \dfrac{\sin x}{\cos x}$, or by defining the tangent of the angle AOB (shown as a 50° angle in the diagram on the right) as the y-coordinate of the point where OA, continued as necessary, meets the vertical line through B, we obtain the graph of $y = \tan x$ as that shown below, for $0 \le x \le 180°$





Continuing this graph to left and right we obtain the graph of $y = \tan x$, for $-2\pi \le x \le 4\pi$, as shown below.



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Note the following:

- Though the previous graph is for $-2\pi \le x \le 4\pi$ the reader should consider the graph of $y = \tan x$ continuing indefinitely to the left and right.
- The graph repeats itself every π radians (or 180°).
  The period of the graph is π radians (or 180°). Thus $\tan(x \pm \pi) = \tan x$.
  The graph performs one cycle in π radians (or 180°).
- The term 'amplitude' is meaningless when applied to $y = \tan x$.
- The graph is such that $\tan(-x) = -\tan x$.
  (The tangent function is an *odd* function.)

Transformations of $y = \sin x$ (and of $y = \cos x$ and $y = \tan x$)

The graph of $y = a \sin x$ will be that of $y = \sin x$ dilated $\updownarrow$ scale factor 'a' (and reflected in the x-axis if a is negative).

Hence $y = a \sin x$ will have amplitude $|a|$.

The graph of $y = \sin bx$ will be that of $y = \sin x$, dilated $\leftrightarrow$ scale factor $\frac{1}{b}$ (and reflected in the y-axis if b is negative).

Hence whilst $y = \sin x$ performs one cycle in 2π radians, $y = \sin bx$ will perform 'b' cycles in 2π radians.

Thus whilst $y = \sin x$ has a period of 2π radians, $y = \sin bx$ has a period of $\frac{2\pi}{b}$ radians.

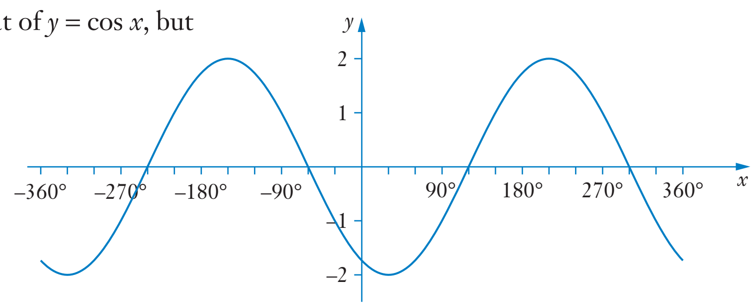
The graph of $y = \sin[b(x + c)]$ will be that of $y = \sin bx$ moved left 'c' units (a **phase** shift).

The graph of $y = \sin(x) + d$ will be that of $y = \sin x$ moved up 'd' units.

The graph on the right looks like that of $y = \cos x$, but

- stretched $\updownarrow$ scale factor 2
- reflected in the x-axis
- not stretched $\leftrightarrow$
- moved $\rightarrow$ 30°

to give its equation as
$y = -2\cos(x - 30°)$.



Alternatively we could consider it as looking to be that of $y = \sin x$

- stretched $\updownarrow$ scale factor 2
- and moved $\rightarrow$ 120°

to give its equation as $y = 2\sin(x - 120°)$.

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Angle sum and angle difference identities

The fact that on the previous page we obtained two apparently different equations for the one graph should not have troubled you if you remembered the existence of the angle sum and angle difference identities:

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

Considering again the two equations found for the same graph, i.e. $y = -2\cos(x - 30°)$ and $y = 2\sin(x - 120°)$:

$$\begin{aligned} y &= -2\cos(x - 30°) \\ &= -2(\cos x \cos 30° + \sin x \sin 30°) \\ &= -2\left(\cos x \times \frac{\sqrt{3}}{2} + \sin x \times \frac{1}{2}\right) \\ &= -\sqrt{3}\cos x - \sin x \end{aligned}$$

$$\begin{aligned} y &= 2\sin(x - 120°) \\ &= 2(\sin x \cos 120° - \cos x \sin 120°) \\ &= 2\left(\sin x \times \left(-\frac{1}{2}\right) - \cos x \times \frac{\sqrt{3}}{2}\right) \\ &= -\sin x - \sqrt{3}\cos x, \\ &= -\sqrt{3}\cos x - \sin x \text{ as before.} \end{aligned}$$

Note that the above expansions also assume that you are familiar with **exact values**.

The next two pages remind you of the proof of the angle sum and angle difference identities and make use of another identity it will also be assumed you are familiar with, namely the **Pythagorean identity**:

$$\sin^2\theta + \cos^2\theta = 1$$

The proofs also assume you are familiar with determining the distance between two points of known coordinates.

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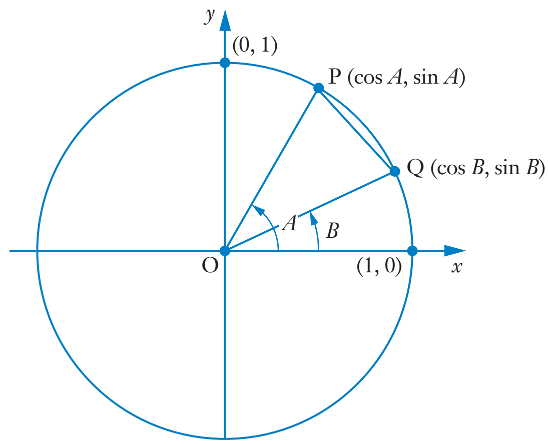
Proof of angle sum and angle difference identities

Consider the points P and Q lying on the unit circle as shown in the diagram on the right.

From our unit circle definition of sine and cosine the coordinates of P and Q will be as shown.



The length of the line joining two points can be found by determining the following:

$$\sqrt{(\text{change in the } x\text{-coordinates})^2 + (\text{change in the } y\text{-coordinates})^2}$$

Thus $\quad \text{PQ} = \sqrt{(\cos A - \cos B)^2 + (\sin A - \sin B)^2}$

$$= \sqrt{\cos^2 A - 2\cos A \cos B + \cos^2 B + \sin^2 A - 2\sin A \sin B + \sin^2 B}$$

$$= \sqrt{\cos^2 A + \sin^2 A + \sin^2 B + \cos^2 B - 2\cos A \cos B - 2\sin A \sin B}$$

$$= \sqrt{1 + 1 - 2(\cos A \cos B + \sin A \sin B)}$$

$$= \sqrt{2 - 2(\cos A \cos B + \sin A \sin B)} \qquad [\text{I}]$$

However, if instead we apply the cosine rule to triangle OPQ:

$$\text{PQ} = \sqrt{1^2 + 1^2 - 2(1)(1)\cos(A - B)}$$

$$= \sqrt{2 - 2\cos(A - B)} \qquad [\text{II}]$$

Comparing [I] and [II] we see that

$$\cos(A - B) = \cos A \cos B + \sin A \sin B \qquad [1]$$

Replacing B by $(-B)$, and remembering that $\cos(-B) = \cos B$ and $\sin(-B) = -\sin B$, it follows that

$$\cos(A - (-B)) = \cos A \cos(-B) + \sin A \sin(-B)$$
$$= \cos A \cos B - \sin A \sin B$$

i.e. $\quad \cos(A + B) = \cos A \cos B - \sin A \sin B \qquad [2]$

From [1], $$\cos\left(\frac{\pi}{2}-\theta\right) = \cos\frac{\pi}{2}\cos\theta + \sin\frac{\pi}{2}\sin\theta$$
$$= (\,0\,)\cos\theta + (\,1\,)\sin\theta$$
$$= \sin\theta$$

Replacing $\frac{\pi}{2} - \theta$ by ϕ (and hence θ by $\frac{\pi}{2} - \phi$) it follows that

$$\cos\phi = \sin\left(\frac{\pi}{2}-\phi\right)$$

Thus $\boxed{\cos\left(\frac{\pi}{2}-A\right) = \sin A}$ and $\boxed{\sin\left(\frac{\pi}{2}-A\right) = \cos A.}$

We can now use these facts to determine expansions for $\sin(A+B)$ and for $\sin(A-B)$.

$$\begin{aligned}\sin(A-B) &= \cos[\,90° - (A-B)\,] \\ &= \cos(90° - A + B) \\ &= \cos(90° - A)\cos B - \sin(90° - A)\sin B \\ &= \sin A\cos B - \cos A\sin B\end{aligned}$$

i.e. $\boxed{\sin(A-B) = \sin A\cos B - \cos A\sin B}$ [3]

Replacing B by $(-B)$ in [3] gives:

$$\begin{aligned}\sin(A-(-B)) &= \sin A\cos(-B) - \cos A\sin(-B) \\ &= \sin A\cos B + \cos A\sin B\end{aligned}$$

i.e. $\boxed{\sin(A+B) = \sin A\cos B + \cos A\sin B}$

Now: $$\tan(A \pm B) = \frac{\sin(A\pm B)}{\cos(A\pm B)}$$
$$= \frac{\sin A\cos B \pm \cos A\sin B}{\cos A\cos B \mp \sin A\sin B}$$
$$= \frac{\dfrac{\sin A\cos B}{\cos A\cos B} \pm \dfrac{\cos A\sin B}{\cos A\cos B}}{\dfrac{\cos A\cos B}{\cos A\cos B} \mp \dfrac{\sin A\sin B}{\cos A\cos B}}$$

Thus $\boxed{\tan(A\pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A\tan B}}$

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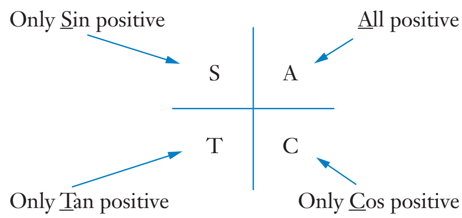
Solving trigonometric equations

The diagram on the right summarises the positive and negative nature of sine, cosine and tangent in the various quadrants.



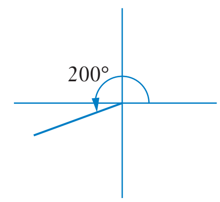
WS Calculating exact values of trigonometric functions

WS Solving periodic functions

Using this awareness of 'what's positive where' we can express the sine (or cosine or tangent) of any angle as the sine (or cosine or tangent) of the acute angle made with the x-axis, together with the appropriate sign.

For example, consider sin 200°.

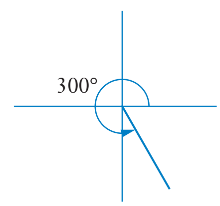
As the diagram on the right shows, an angle of 200° makes 20° with the x-axis and lies in the 3rd quadrant, where the sine function is negative.



Thus sin 200° = –sin 20°

Now consider cos (300°).

An angle of 300° makes 60° with the x-axis and lies in the 4th quadrant, where the cosine function is positive.



Thus $\cos 300° = \cos 60°$
$\qquad\qquad\quad = 0.5$

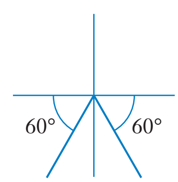
These ideas can help when we have to solve equations involving trigonometric functions.

Suppose we are asked to solve $\sin x = -\frac{\sqrt{3}}{2}$ for $0 \le x \le 360°$.

With the sine being negative solutions must lie in the 3rd and 4th quadrants.

From our exact values we know that $\sin 60° = \frac{\sqrt{3}}{2}$.

Thus the solutions make 60° with the x-axis as shown diagrammatically on the right.



Using this diagram to obtain solutions in the required interval we have

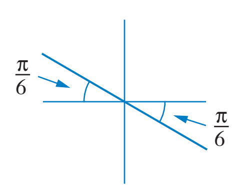
$$x = 240°, 300°.$$

Suppose now that we are asked to solve $\tan x = -\frac{1}{\sqrt{3}}$ for $-\pi \le x \le \pi$.

With the tangent being negative solutions must lie in the 2nd and 4th quadrants.

From our exact values we know that $\tan \frac{\pi}{6} = \frac{1}{\sqrt{3}}$.

Thus the solutions make $\frac{\pi}{6}$ radians with the x-axis as shown diagrammatically on the right.



Using this diagram to obtain solutions in the required interval gives:

$$x = -\frac{\pi}{6}, \frac{5\pi}{6}.$$

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